Scale and conformal invariance for cold atomic gases

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Quantum Science Seminar, March 3, 2022

Scale invariance

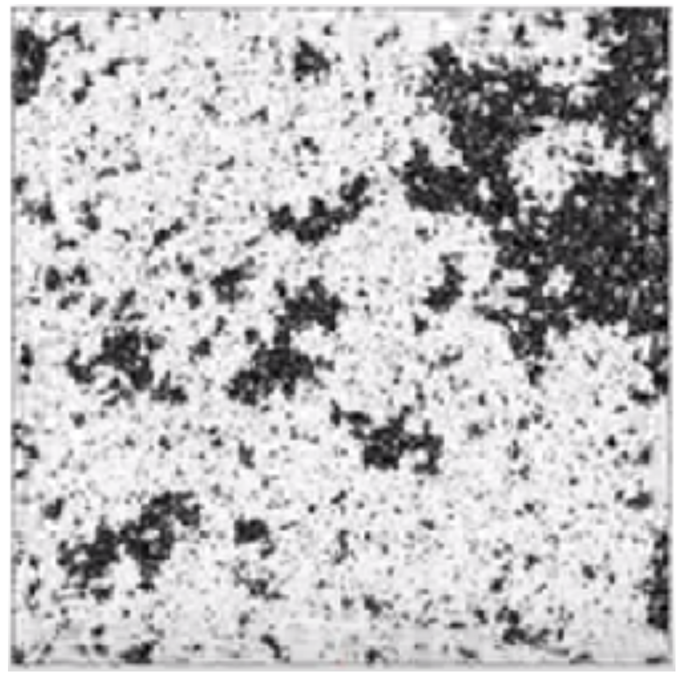
A concept that was introduced in the 70's in high energy physics

Can there be physical systems with no intrinsic energy/length scale?

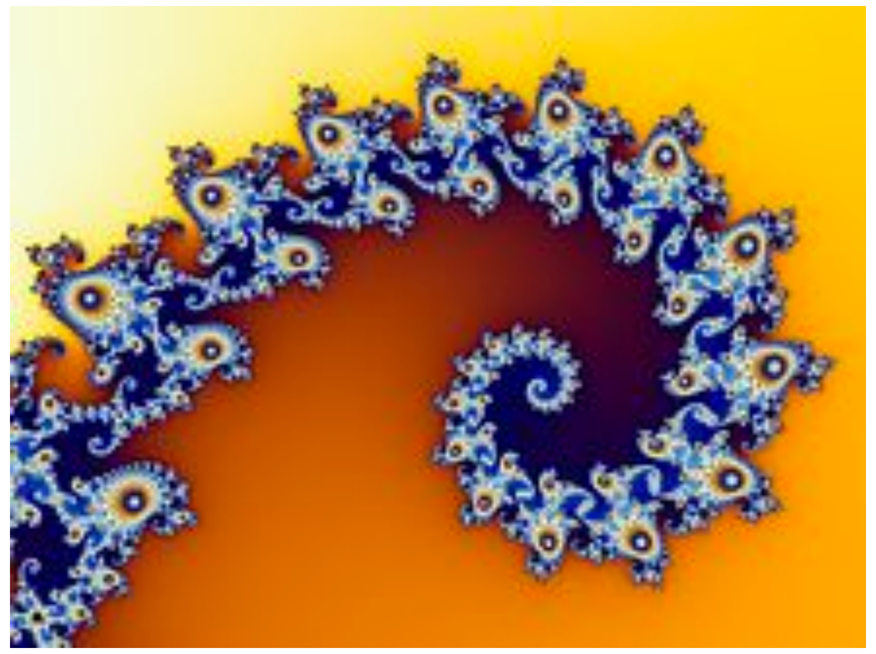
Need to explain the behavior of e^- - nucleon scattering cross-sections

This concept later found many applications in physics, maths, biology, etc.

Phase transitions and renormalization group



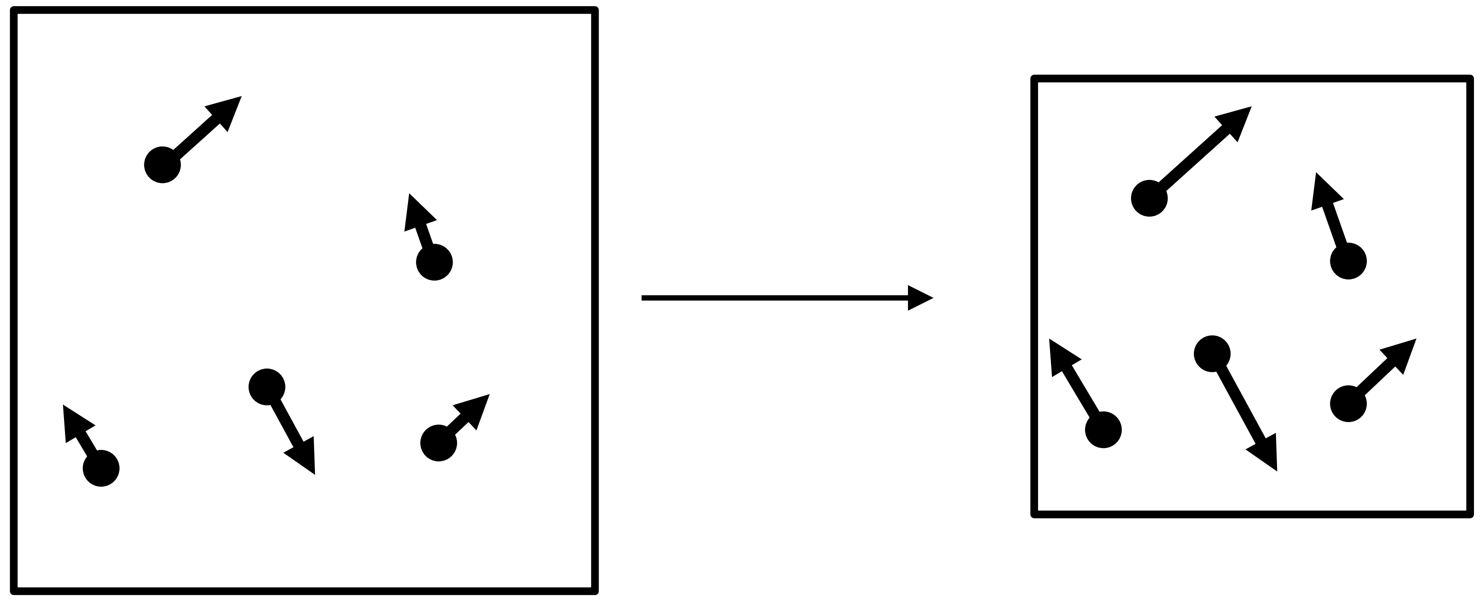
Fractals



Scale invariance in a gas of particles

Consider a fluid whose equations of motion, *i.e.* its action $\int E\, \mathrm{d}t$, are invariant in the following rescaling:

Positions: $\boldsymbol{r} \to \boldsymbol{r}/\lambda$ Time: $t \to t/\lambda^2$



Velocity: $\boldsymbol{v} \to \lambda \boldsymbol{v}$

Considerable simplification of the study of equilibrium properties and dynamics

Clearly $E_{\text{kin}} \to \lambda^2 E_{\text{kin}}$, implying that $\int E_{\text{kin}}\, \mathrm{d}t$ is invariant

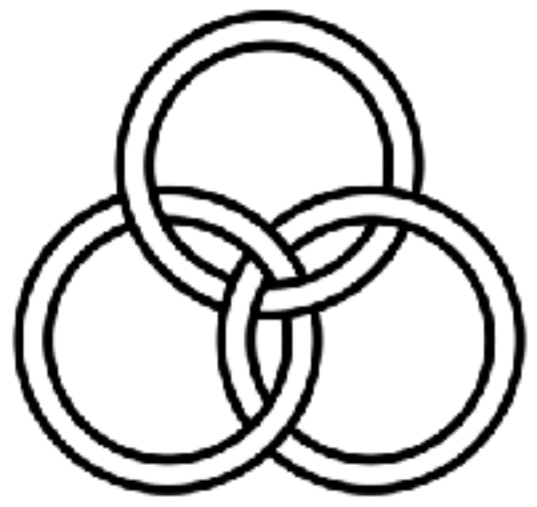
What about interactions? Can we achieve $E_{\text{int}} \to \lambda^2 E_{\text{int}}$ when $\boldsymbol{r} \to \boldsymbol{r}/\lambda$?

Gases with scale invariant interactions

$\boldsymbol{r} \to \boldsymbol{r}/\lambda \qquad E_{\text{int}} \to \lambda^2 E_{\text{int}}$

The simplest case: the $1/r^2$ potential $\quad V = \sum_{i<j} \frac{g}{r_{ij}^2}$

Calogero-Moser-Sutherland model in 1D



Efimov problem in 3D

For such a potential, there is no length scale associated to interactions

Reminder: for a power-law potential g/r^n, the relevant (quantum!) length scale ℓ is obtained by equating kinetic and potential energy

$$\frac{\hbar^2}{m\ell^2} = \frac{g}{\ell^n}$$

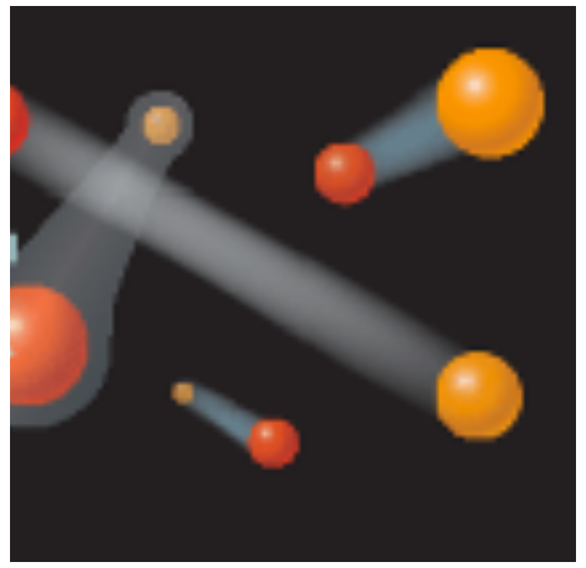
- Coulomb interaction ($n = 1,\ g = e^2$): ℓ = Bohr radius $\hbar^2/me^2$
- Van der Waals interaction ($n = 6,\ g = C_6$): ℓ = van der Waals radius $\propto \left(mC_6/\hbar^2\right)^{1/4}$

No characteristic length ℓ for $n = 2$!

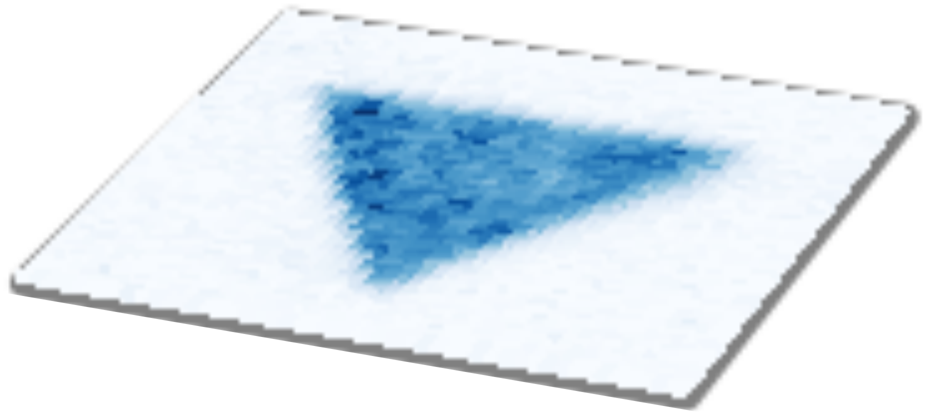
Cold atomic gases with scale invariant interactions

$$\boldsymbol{r} \to \boldsymbol{r}/\lambda \qquad\qquad E_{\rm int} \to \lambda^2 E_{\rm int}$$

- 3D spin 1/2 Fermi gas in the unitary regime: infinite scattering length, hence no length scale associated to interactions



- Contact interaction in a 2D Bose gas:



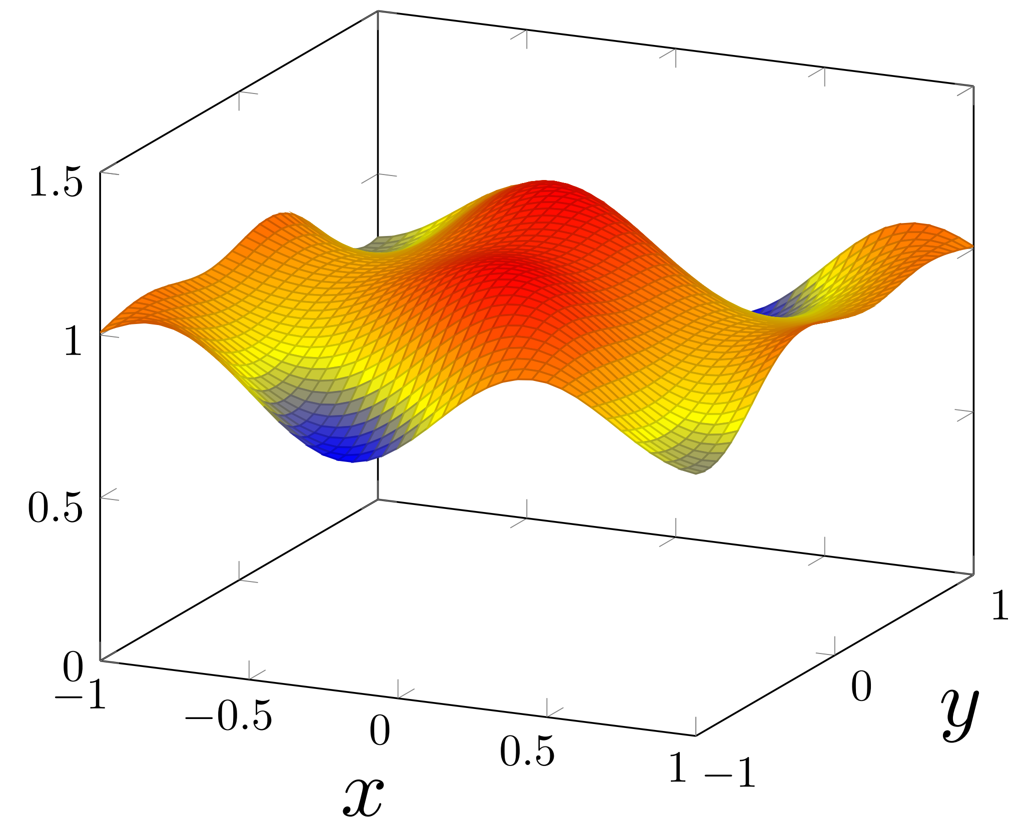
$$\boldsymbol{r} \to \boldsymbol{r}/\lambda \qquad\qquad g\,\delta(\boldsymbol{r}) \to g\,\delta(\boldsymbol{r}/\lambda) = \lambda^2\; g\,\delta(\boldsymbol{r})$$

Valid only for relatively weak interactions, so that a classical field description (Gross-Pitaevskii equation) is valid (otherwise, ***quantum anomaly*** from the regularisation of $\delta(\boldsymbol{r})$)

Classical field approach to the 2D Bose gas

Describe the gas by a classical field $\psi(\boldsymbol{r}, t)$ obeying the Gross-Pitaevskii equation

Energy of the gas: $E(\psi) = E_{\mathrm{kin}}(\psi) + E_{\mathrm{int}}(\psi)$



$$E_{\mathrm{kin}}(\psi) = \frac{\hbar^2}{2m} \int |\boldsymbol{\nabla}\psi|^2 \qquad E_{\mathrm{int}}(\psi) = \frac{\hbar^2}{2m}\tilde{g} \int |\psi|^4$$

$\tilde{g}$: interaction strength

No singularity at the classical field level

In 3D, $\tilde{g} = 4\pi a$ where a is the scattering length

In 2D, the interaction strength $\tilde{g}$ is dimensionless: no length scale associated with interactions

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Time-independent problems

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Solitons in 2D

The Efimov effect

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Conformal invariance and the SO(2,1) dynamical symmetry

The breathing mode

Breathers

Scale-invariant equation of state

For a “standard” cold 3D gas, the scattering length a brings the energy scale $\epsilon \equiv \hbar^2/ma^2$

Exemple of an equation of state: $n\lambda^3 = \mathscr{F}\left(\frac{k_B T}{\epsilon}, \frac{\mu}{\epsilon}\right)$ *i.e., a 2-variable function*

For a scale-invariant Fermi gas ($a = 0$ or $a = \infty$), it must read $n\lambda^3 = \mathscr{G}\left(\frac{\mu}{k_B T}\right)$

Considerable simplification (1-variable function) which leads to $PV = \frac{2}{3}E$ T.L. Ho, 2004

Similarly for a 2D Bose gas: $n\lambda^2 = \mathscr{G}\left(\frac{\mu}{k_B T}, \tilde{g}\right)$ $\tilde{g}$ dimensionless coupling

$\longrightarrow$ $PV = E$

The equation of state of the 2D Bose gas

$$n\lambda^2 = \mathscr{G}\left(\frac{\mu}{k_B T}, \tilde{g}\right)$$

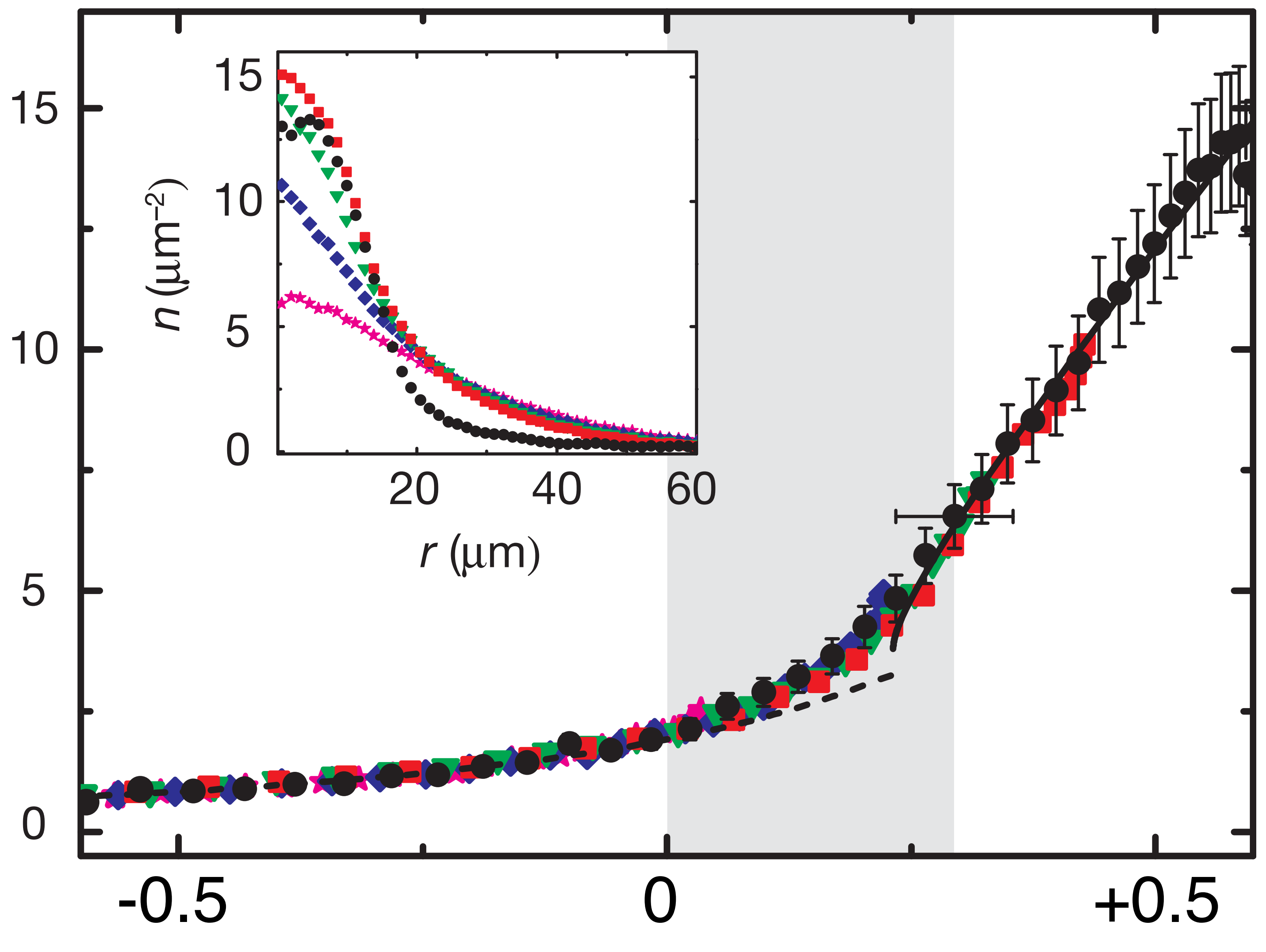
Theory using a classical-field analysis: Prokof'ev & Svistunov

Measurements : Chicago, Paris, Cambridge

Smooth external trapping potential $V_{\text{trap}}(r)$ + local-density approximation: $\mu(r) = \mu(0) - V_{\text{trap}}(r)$

A single image gives access to the desired function $\mathscr{G}$: $n(r)\,\lambda^2 = \mathscr{G}\left(\frac{\mu(r)}{k_B T}\right)$

Phase-space density $n\lambda^2$



$\mu/k_B T$

All density profiles obtained for various atom numbers and various temperatures collapse on the same universal curve (for a given interaction strength, here $\tilde{g} = 0.26$)

from Hung et al, Nature **470**, 236 (2011)

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- ➡ Solitons in 2D
- The Efimov effect

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Solitons for the Gross-Pitaevskii equation

Look for a stationary wave function ψ solution of the variational problem $\delta\left[E(\psi)\right] = 0$ for an attractive non-linearity $g < 0$

$$E[\psi] = \frac{1}{2}\int\left(\left|\nabla\psi\right|^2 + g\left|\psi\right|^4\right)\,\mathrm{d}^D r$$

$\hbar = m = 1$

$\int |\psi|^2 = N$

Relevant in optics, atomic physics, condensed matter…

Dimensional analysis for a wave packet of size ℓ: $\quad \frac{E(\ell)}{N} \sim \frac{1}{\ell^2} - \frac{N|g|}{\ell^D}$

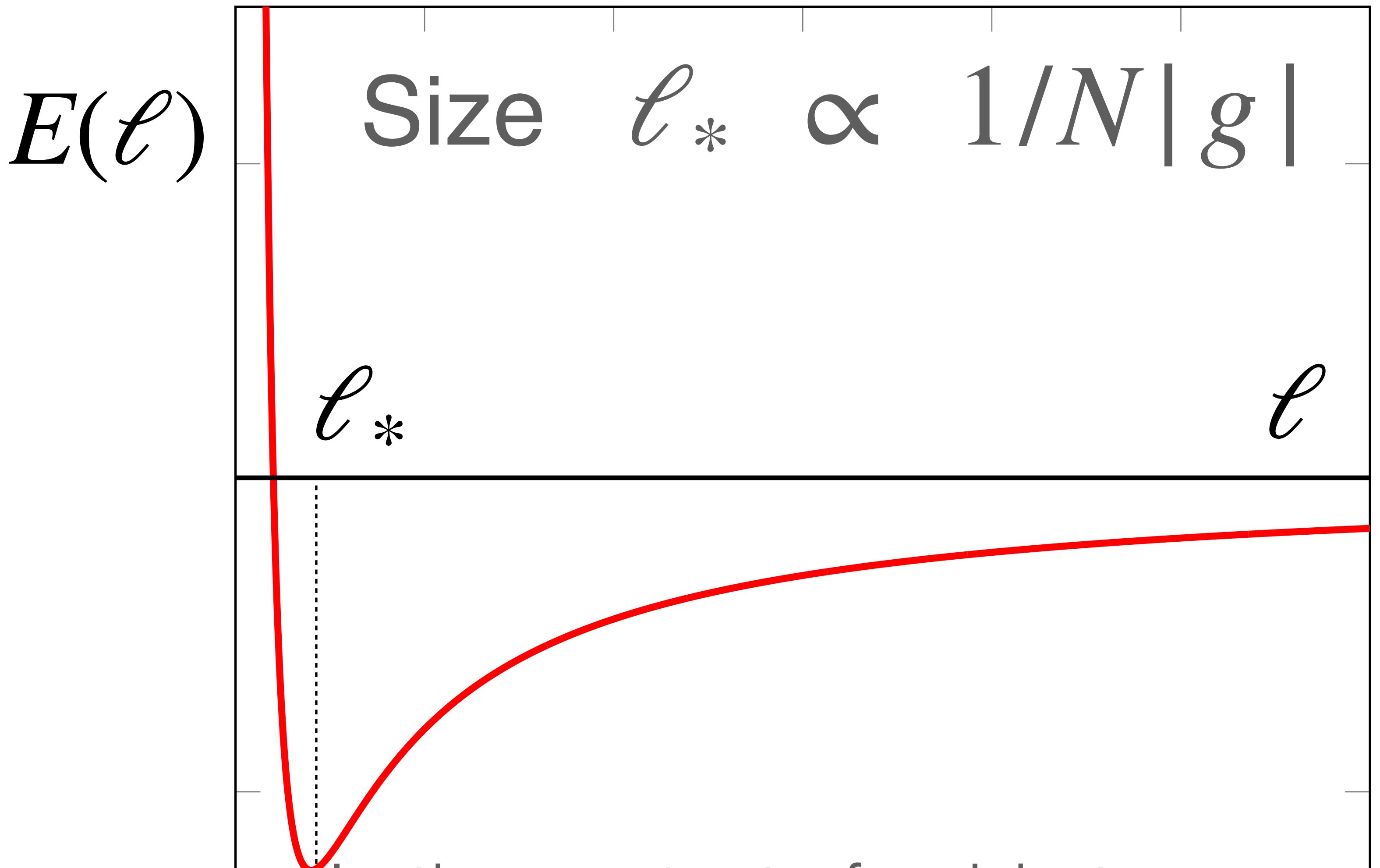
Crucial role of dimensionality D

Solitons in 1D, 2D, 3D

Wave packet of size ℓ in dimension D :

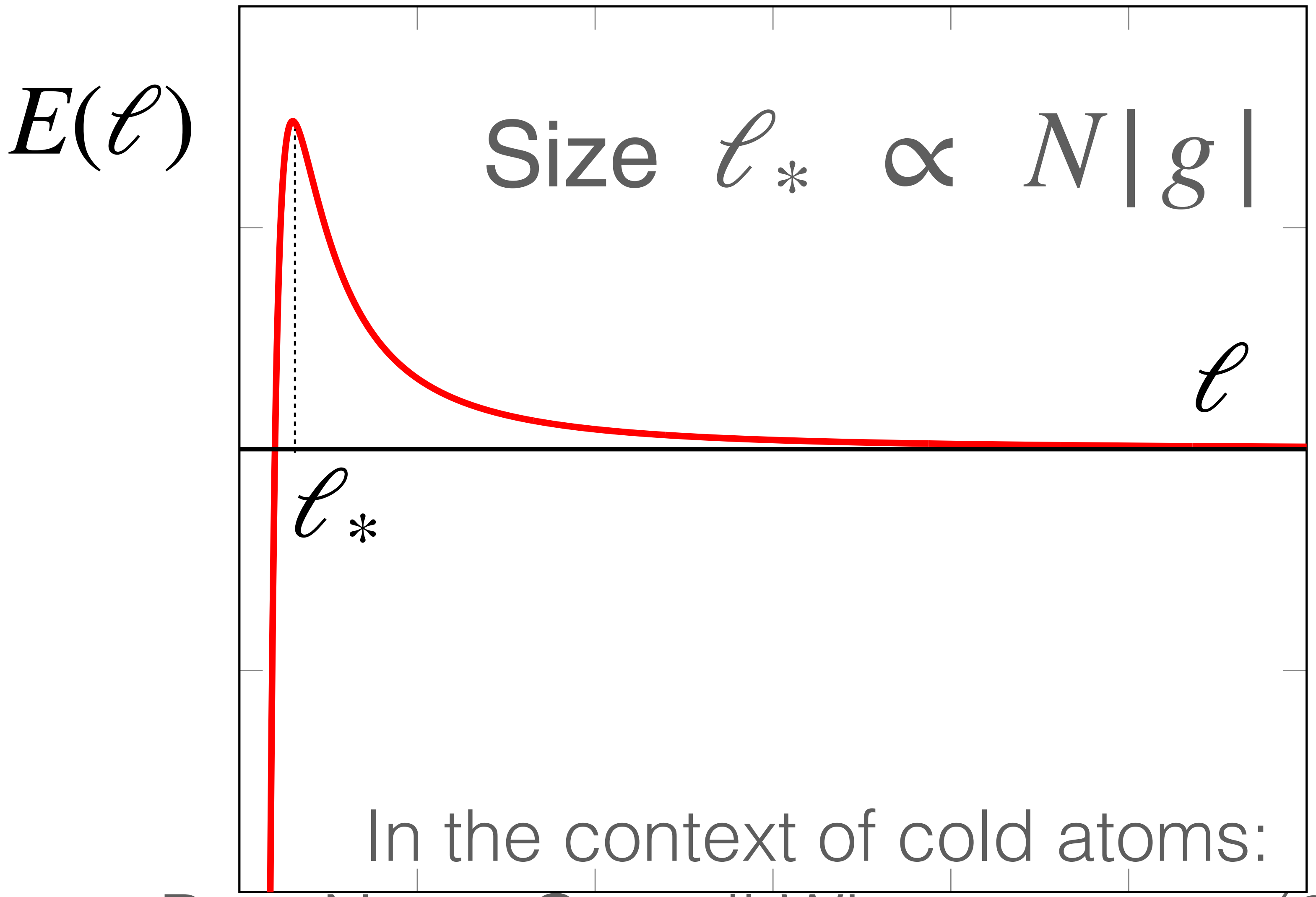
$$\frac{E(\ell)}{N} \sim \frac{1}{\ell^2} - \frac{N|g|}{\ell^D}$$

In 1D: Stable solution for any N and any g



In the context of cold atoms:
Salomon and Hulet's groups (2002)

In 3D: Dynamically unstable extremum



In the context of cold atoms:
BoseNova: Cornell-Wieman group (2001)

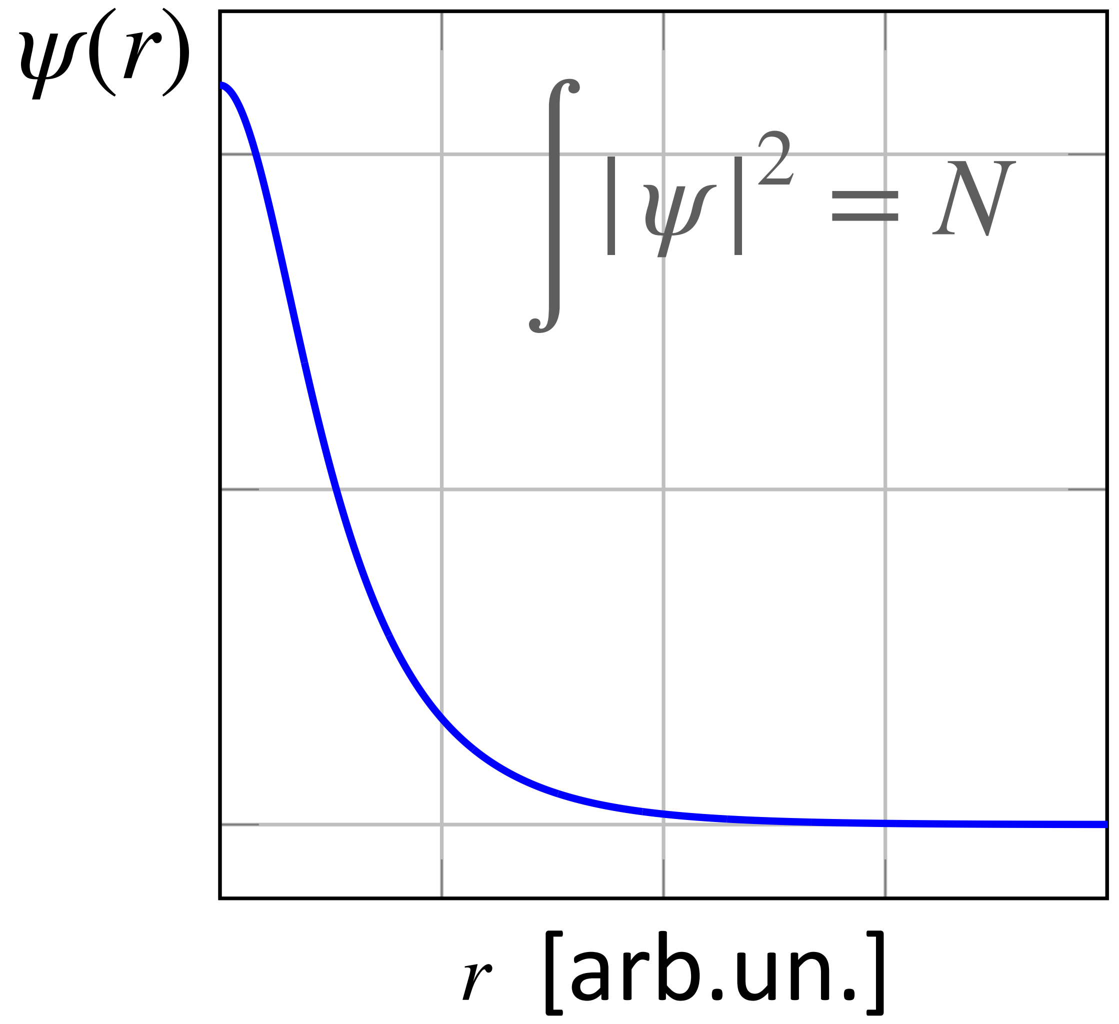
2D is a critical dimension: Stationary solutions can be obtained only for discrete values of $N|g|$

2D: the Townes soliton

$$E[\psi] = \frac{1}{2}\int \left(\left|\nabla\psi\right|^2 + g\left|\psi\right|^4 \right) \mathrm{d}^2 r$$

Chiao, Garmire & Townes, 1964

Radially symmetric, node-less solution of $-\frac{1}{2}\nabla^2\psi + g\,\psi^3 = \mu\,\psi$



Such a solution exists only if $(Ng)_{\text{Townes}} = -\,5.85...$

It has $E = 0$ and $\mu < 0$

Once a particular solution is known, scale invariance provides a continuous family of solutions

$\phi(\boldsymbol{r}) = \lambda\,\psi(\lambda\boldsymbol{r})$ $\qquad \mu_\phi = \lambda^2\mu$ $\qquad \lambda$ real

No particular length scale for the Townes soliton when it exists

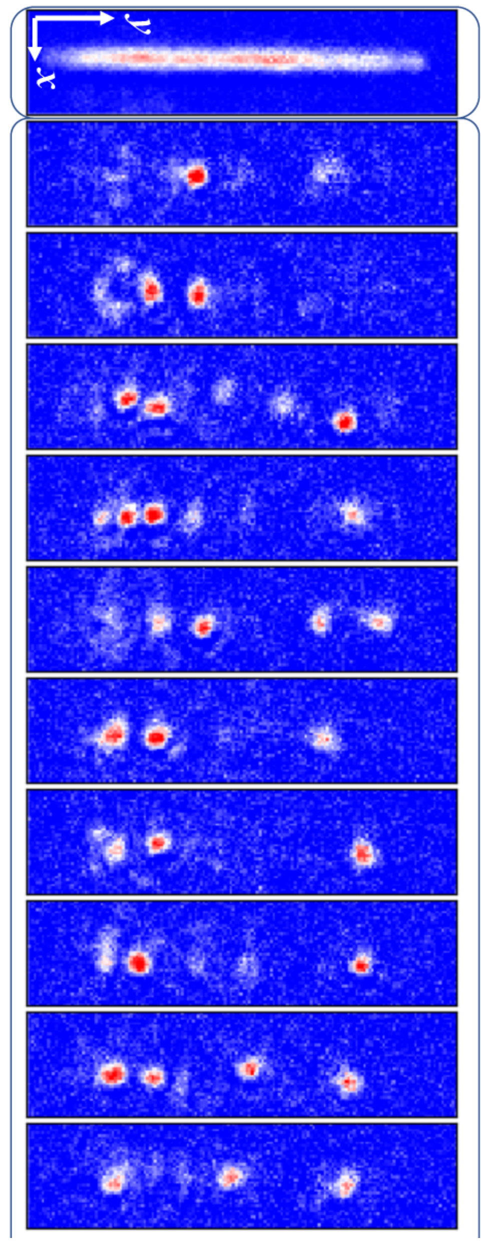
However: Instable with respect to a change in shape or in Ng

Observation of Townes soliton with cold atomic gases

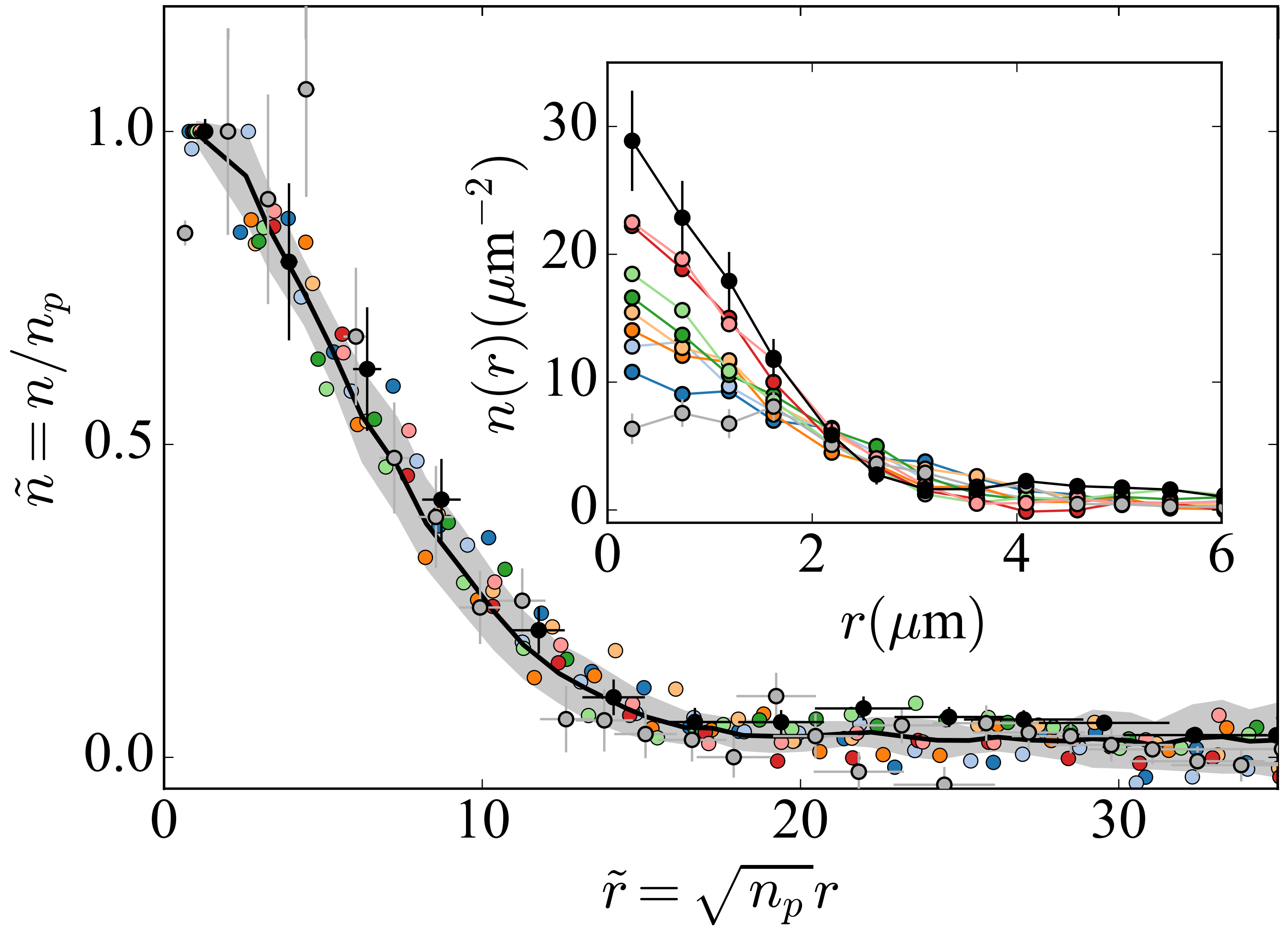
One needs to achieve an effective attractive interaction $\tilde{g} < 0$

Paris group: Phys. Rev. Lett. 127, 023603 (2021), use of a two-component gas with ^{87}Rb
Purdue group: Phys. Rev. Lett. 127, 023604 (2021), use of a Feshbach resonance with ^{133}Cs

Quench $\tilde{g} : +0.13 \rightarrow -0.0215$
and switch from 1D to 2D



Rescale all “droplets” together



Atom number/droplet: $\langle N\tilde{g}\rangle = -6.0\,(8)$

to be compared with $(N\tilde{g})_{\text{Townes}} = -5.85...$

from Chen & Hung, PRL **127**, 023604 (2021)

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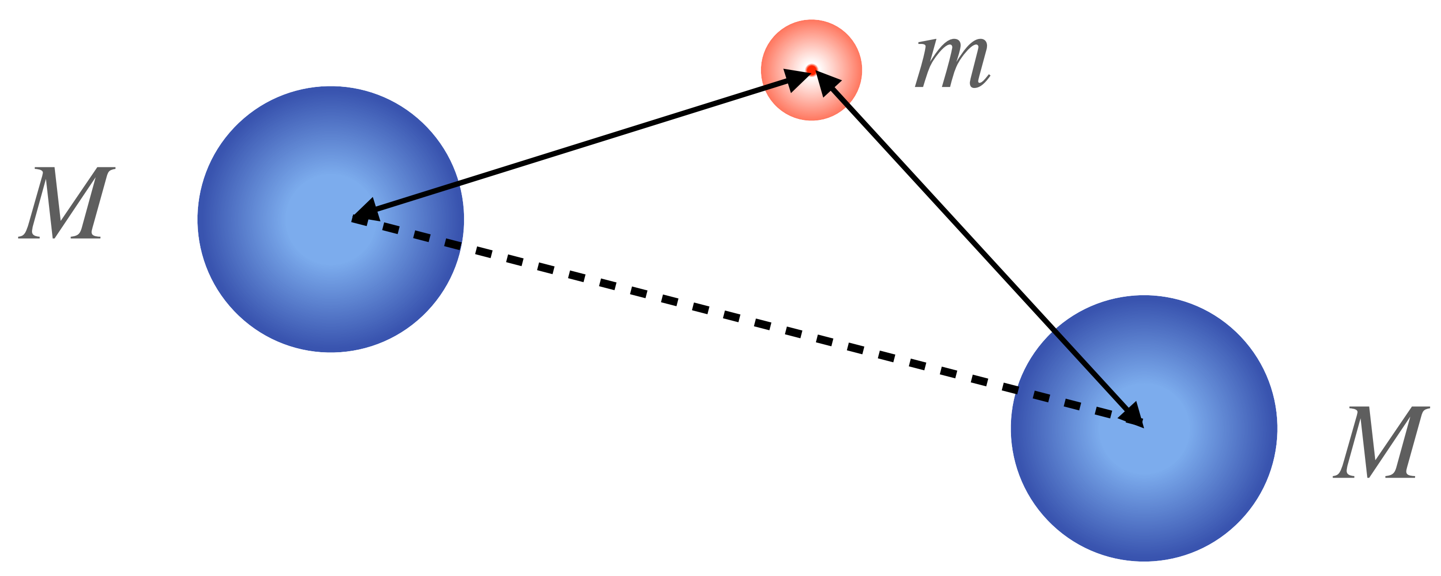
➡ The Efimov effect   Efimov, 1970



Fonseca et al, 1979

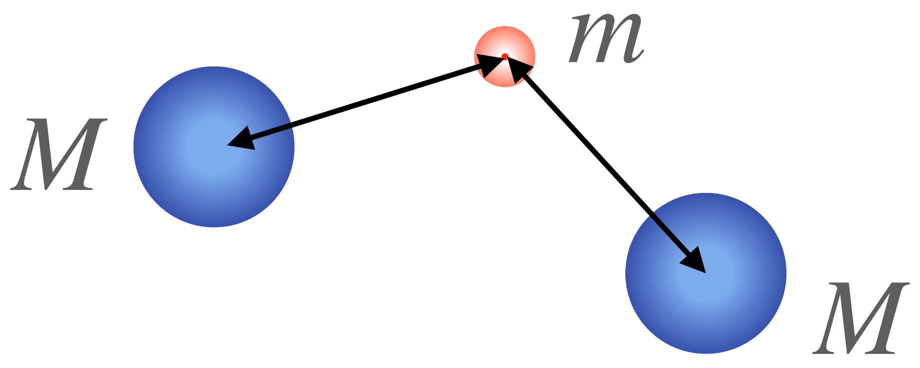
We look here at the 3-body problem “Heavy + Heavy + Light”

→ No direct interaction “Heavy-Heavy”

→ Heavy-Light contact interaction with scattering length a

Limit $a \to \infty$: no two-body bound state “Heavy + Light”

Emergence of $1/R^2$ potential between the heavy particles



Born-Oppenheimer approach: assume first the heavy particles fixed in $\pm \boldsymbol{R}/2$

⟶ Wave function for the light particle for the energy $\varepsilon = -\hbar^2\kappa^2/2m$: $\psi(\boldsymbol{r}) \propto \dfrac{e^{-\kappa|\boldsymbol{r}-\boldsymbol{R}/2|}}{|\boldsymbol{r}-\boldsymbol{R}/2|} + \dfrac{e^{-\kappa|\boldsymbol{r}+\boldsymbol{R}/2|}}{|\boldsymbol{r}+\boldsymbol{R}/2|}$

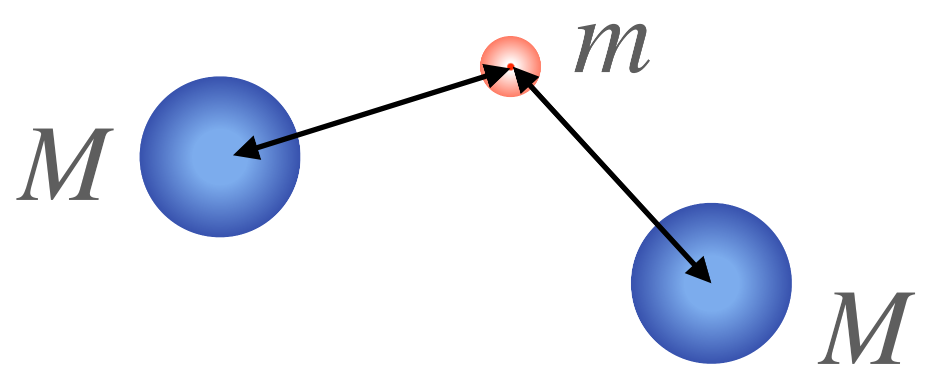
⟶ Bethe-Peierls boundary condition for each "Heavy-Light" contact interaction:

$$\left.\frac{\partial(r\psi)}{\partial r}\right|_{r=0} = -\frac{1}{a}\,(r\psi)_{r=0} \quad\Rightarrow\quad \kappa - \frac{e^{-\kappa R}}{R} = \frac{1}{a}$$

⟶ When $a \to \infty$, the solution of $\kappa R = e^{-\kappa R}$ is $\kappa R \approx 0.57$, hence $\kappa \approx \dfrac{0.57}{R}$ and $\varepsilon(R) \approx -0.32\,\dfrac{\hbar^2}{2mR^2}$

Then, the energy $\varepsilon(R)$ of the ground state of the light particle plays the role of a potential energy for the motion of the two heavy ones: attractive $1/R^2$ potential

Bound states of the 3-body system



Motion of the heavy particles in the $1/R^2$ potential created by the heavy-light resonant interaction

$$-\frac{\hbar^2}{M}\nabla^2\Psi(\boldsymbol{R}) - \frac{g}{R^2}\Psi(\boldsymbol{R}) \;=\; E\,\Psi(\boldsymbol{R}) \qquad g = 0.32\,\frac{\hbar^2}{2m}$$

We look for $E < 0$

Scale invariance of g/R^2: if $\Psi(\boldsymbol{R})$ is a solution for energy E, then $\Phi(\boldsymbol{R}) = \Psi(\boldsymbol{R}/\lambda)$ is solution for E/λ^2 .

Continuous spectrum from $E = -\infty$ to $E = 0$?

Need to impose a lower bound E_0, for example by imposing a hard core in $R = R_0$

→ Breaks the continuous scale invariance

→ Keeps a discrete scale invariance: infinite sequence of bound states $E_n = E_0/\lambda^{2n}$ where λ depends on M/m

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From scale to conformal invariance

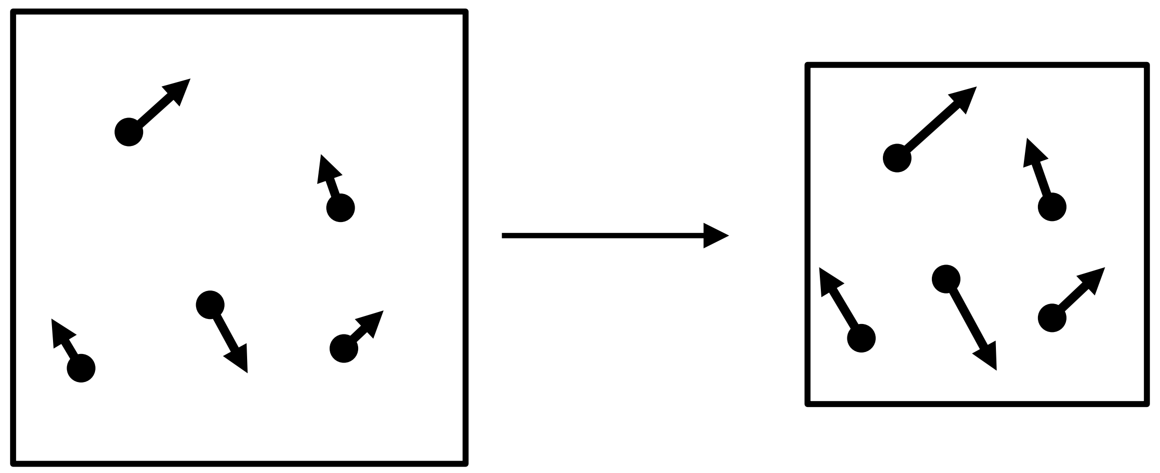
Niederer, 1972-73
Pitaevskii & Rosch, 1997

In addition to the standard Galilean transformations (translations, rotations), there exist 3 types of transformations that leave the unitary 3D Fermi gas or the 2D Bose gas invariant:

Dilatations: $\boldsymbol{r} \to \boldsymbol{r}/\lambda$ $\quad t \to t/\lambda^2$

Time translations: $\boldsymbol{r} \to \boldsymbol{r}$ $\quad t \to t + t_0$

"Expansions": $\boldsymbol{r} \to \dfrac{\boldsymbol{r}}{\gamma t + 1}$ $\quad t \to \dfrac{t}{\gamma t + 1}$



3-parameter group: dynamical symmetry associated with the SO(2,1) two-dimensional Lorentz group

Can be extended to a harmonic trap, with a slight modification of the transformations

The SO(2,1) symmetry in a nutshell

$$\hat{H}_{\mathrm{kin}} = \sum_j \frac{\hat{\boldsymbol{p}}_j^2}{2m} \qquad \hat{H}_{\mathrm{int}} = \frac{1}{2}\sum_{i \neq j} V(\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j) \qquad \hat{H}_{\mathrm{pot}} = \sum_j \frac{1}{2} m\omega^2 \hat{\boldsymbol{r}_j}^2$$

Define the three operators:

$$\begin{cases} \hat{L}_1 = \dfrac{1}{2\hbar\omega}\left(\hat{H}_{\mathrm{kin}} + \hat{H}_{\mathrm{int}} - \hat{H}_{\mathrm{pot}}\right) \\ \hat{L}_2 = \dfrac{1}{4}\sum_j \left(\hat{\boldsymbol{r}_j} \cdot \hat{\boldsymbol{p}_j} + \hat{\boldsymbol{p}_j} \cdot \hat{\boldsymbol{r}_j}\right) \\ \hat{L}_3 = \dfrac{1}{2\hbar\omega}\left(\hat{H}_{\mathrm{kin}} + \hat{H}_{\mathrm{int}} + \hat{H}_{\mathrm{pot}}\right) \end{cases}$$

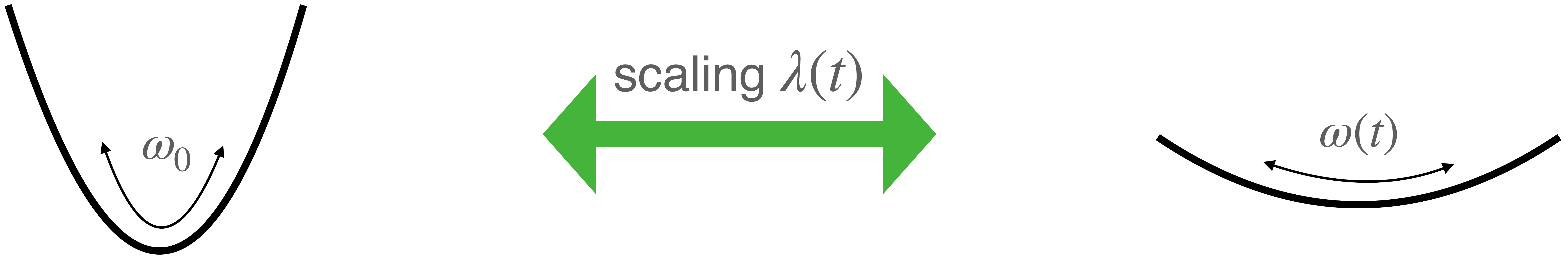
(total Hamiltonian)

Commutation relations: $[\hat{L}_1, \hat{L}_2] = -\mathrm{i}\hbar\hat{L}_3$ $\quad [\hat{L}_2, \hat{L}_3] = \mathrm{i}\hbar\hat{L}_1$ $\quad [\hat{L}_3, \hat{L}_1] = \mathrm{i}\hbar\hat{L}_2$

Close to an angular momentum (SO(3)), but not quite

The invariant is here: $\hat{L}_1^2 + \hat{L}_2^2 - \hat{L}_3^2$

Linking various time-dependent solutions



Conformal invariance allows one to link the solution of the N-body Schrödinger equation in a trap of frequency ω_0 to the solution in a trap with frequency ω, for the same initial state.

ω may possibly depend on time, and even be zero (untrapped case)

The scaling parameter $\lambda(t)$ is the solution of the Ermakov equation: $\frac{\mathrm{d}^2\lambda}{\mathrm{d}t^2} + \omega^2(t)\,\lambda(t) = \frac{\omega_0^2}{\lambda^3(t)}$

Pitaevskii & Rosch, 1997; Kagan et al 1997; Castin & Dum 1997 ;
Castin & Werner, 2004-06 ; Son et al, 2006-07; Nishida & Tan, 2008 ; Gritsev et al, 2010

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Conformal invariance and the SO(2,1) dynamical symmetry

➡ The breathing mode — Pitaevskii & Rosch, 1997

Breathers

A smoking gun of SO(2,1) symmetry: The breathing mode

- Prepare an arbitrary shape for the gas at $t = 0$
- Let the atoms evolve in a 2D harmonic potential of frequency ω in the presence of interactions
- Measure $\langle r^2 \rangle \propto \langle \hat{H}_{\text{pot}} \rangle$ after an evolution time t : Perfectly periodic evolution with frequency 2ω

Direct consequence of the commutation relations, using Heisenberg picture:

$$\hat{L}_1 = \frac{1}{2\hbar\omega}\left(\hat{H}_{\text{kin}} + \hat{H}_{\text{int}} - \hat{H}_{\text{pot}}\right) \qquad \hat{L}_2 = \frac{1}{4}\sum_j \left(\hat{\boldsymbol{r}}_j \cdot \hat{\boldsymbol{p}}_j + \hat{\boldsymbol{p}}_j \cdot \hat{\boldsymbol{r}}_j\right) \qquad \hat{L}_3 = \frac{\hat{H}}{2\hbar\omega}$$

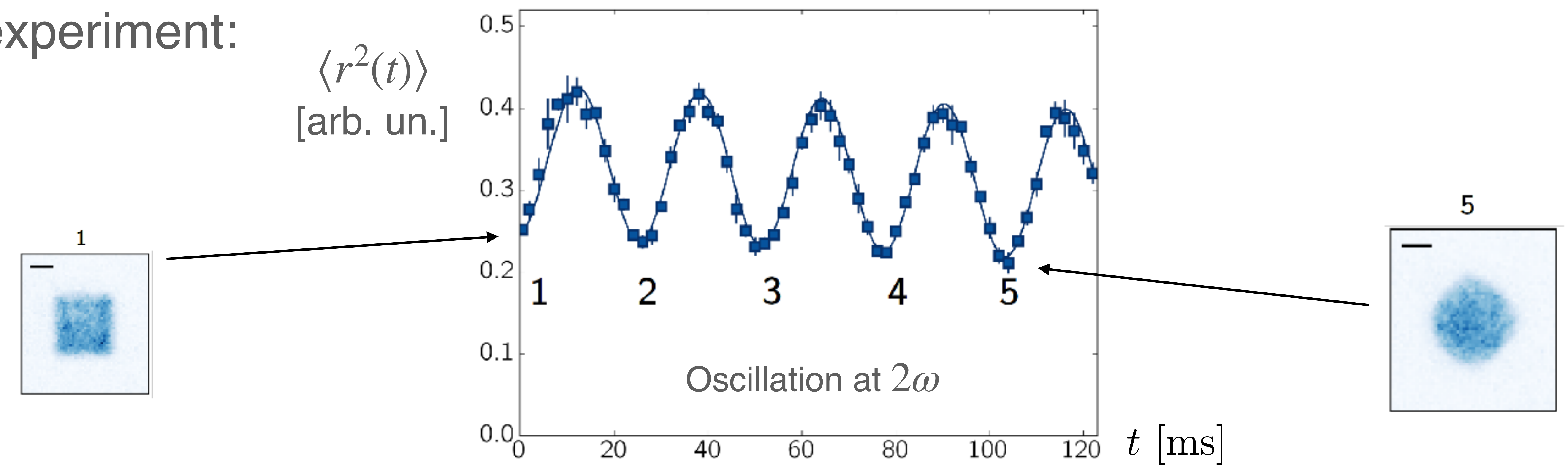
$$\begin{cases} \dfrac{d\hat{L}_1}{dt} = \dfrac{i}{\hbar}[\hat{H}, \hat{L}_1] = -2\omega\,\hat{L}_2 \\[2ex] \dfrac{d\hat{L}_2}{dt} = \dfrac{i}{\hbar}[\hat{H}, \hat{L}_2] = +2\omega\,\hat{L}_1 \end{cases} \longrightarrow \quad \frac{d^2\hat{L}_1}{dt^2} + (2\omega)^2\hat{L}_1 = 0$$

out-of-phase oscillation of $E_{\text{kin}} + E_{\text{int}}$ and E_{pot}

A smoking gun of SO(2,1) symmetry: The breathing mode

- Prepare an arbitrary shape for the gas at $t = 0$
- Let the atoms evolve in a 2D harmonic potential of frequency ω <u>in the presence of interactions</u>
- Measure $\langle r^2 \rangle \propto \langle \hat{H}_{\text{pot}} \rangle$ after an evolution time t : Perfectly periodic evolution with frequency 2ω

A 2D experiment:



Saint-Jalm et al,
Phys. Rev. X 9, 021035 (2019)

→ In 2D, the scale invariance holds only at the classical field level. What about quantum corrections?

→ Are there shapes that lead to a fully periodic motion (i.e. all moments $\langle r^n \rangle$ are periodic) ?

Quantum anomaly for $\langle r^2 \rangle (t)$

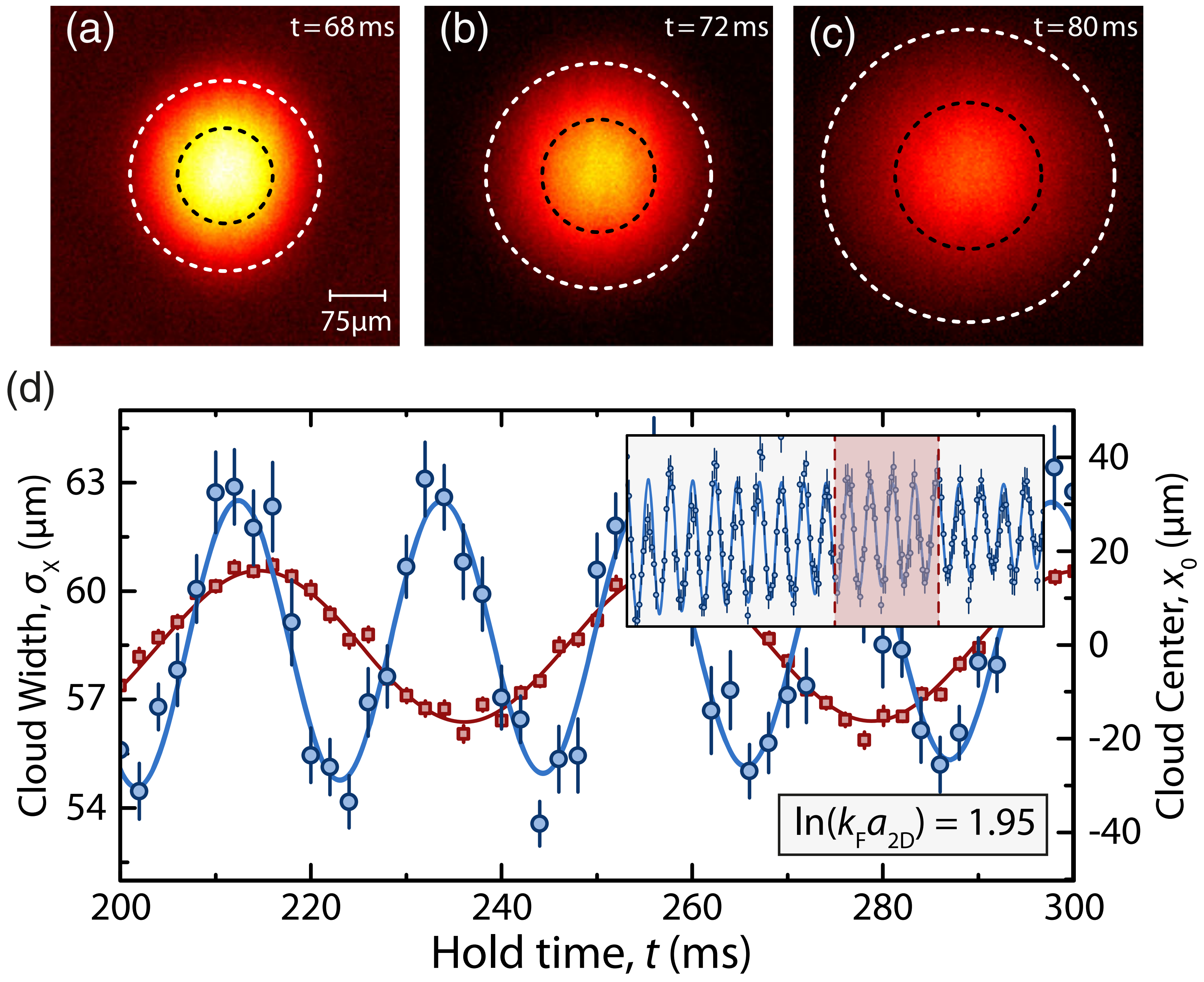
Olshanii, Perrin, Lorent PRL **105**, 095302 (2010)

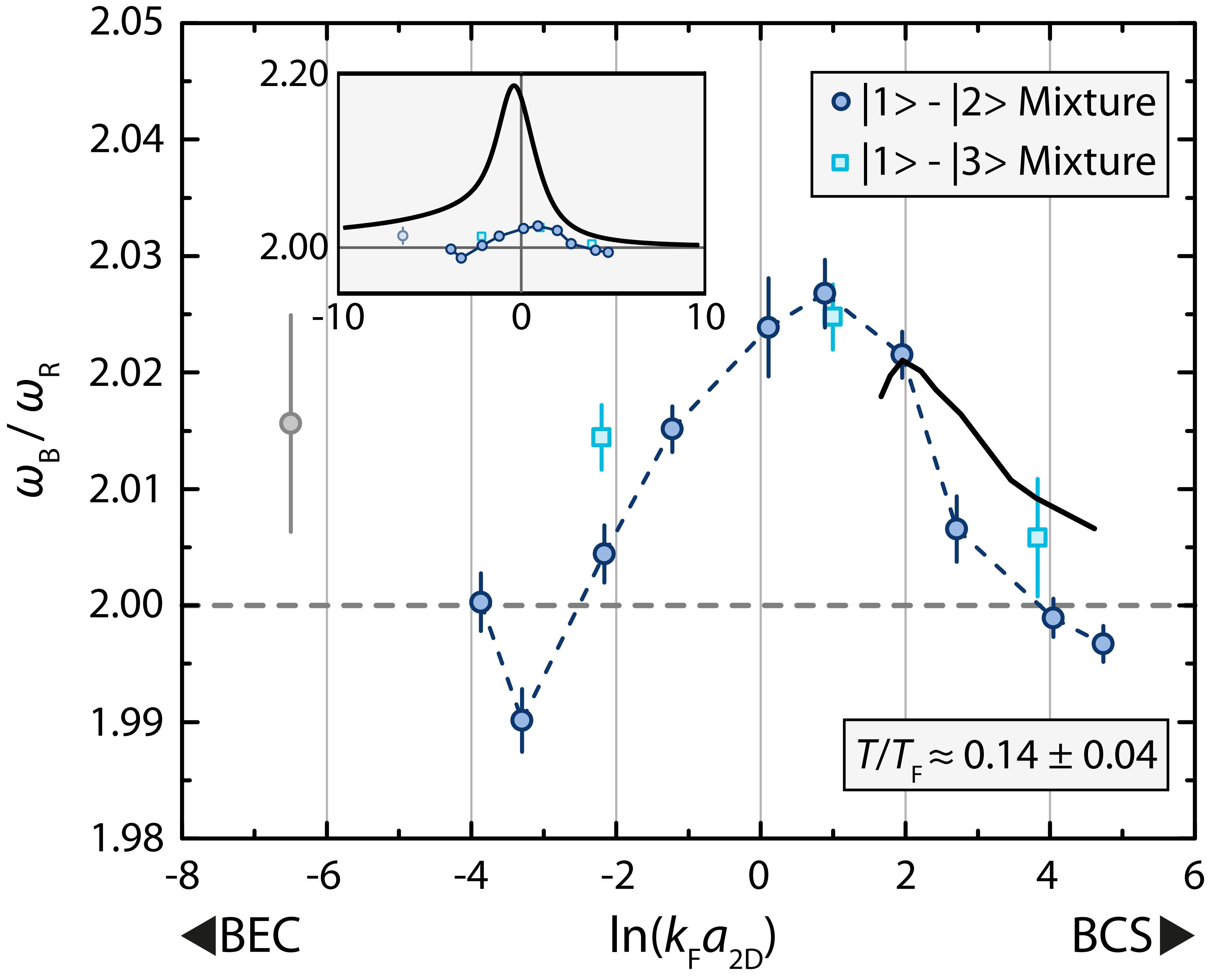
Hofmann, PRL **108**, 185303 (2012)

In 2D, the scale/conformal invariance holds only at the classical field level

The necessary regularization of the $\delta^{(2D)}(\boldsymbol{r}_i - \boldsymbol{r}_j)$ function for a quantum field treatment breaks this symmetry

Recent investigations wit a 2D Fermi gas close to the unitary point:





M. Holten et al,
PRL **121**, 120401 (2018)
[Jochim's group, Heidelberg]

see also T. Peppler et al, PRL **121**, 120402 (2018) [Vale's group, Swinburne]

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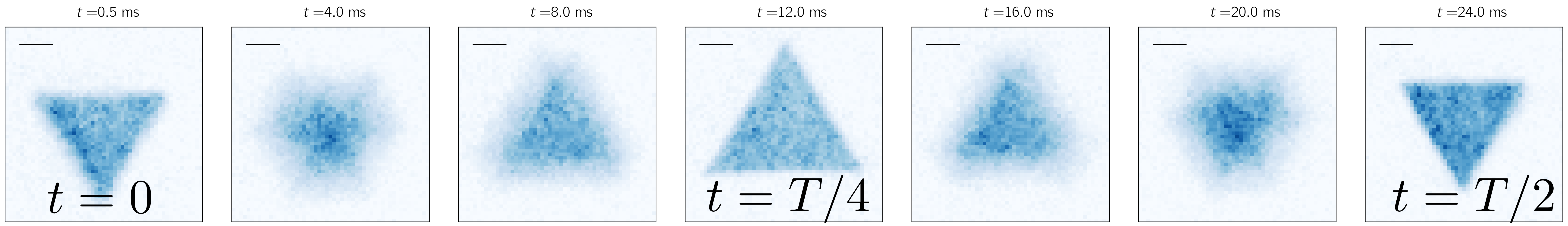
➡ Breathers

Are there shapes that lead to a fully periodic motion at 2ω (i.e. all moments $\langle r^n \rangle$ are periodic) ?

The equilateral triangle in the hydrodynamic ($Ng \gg 1$) regime

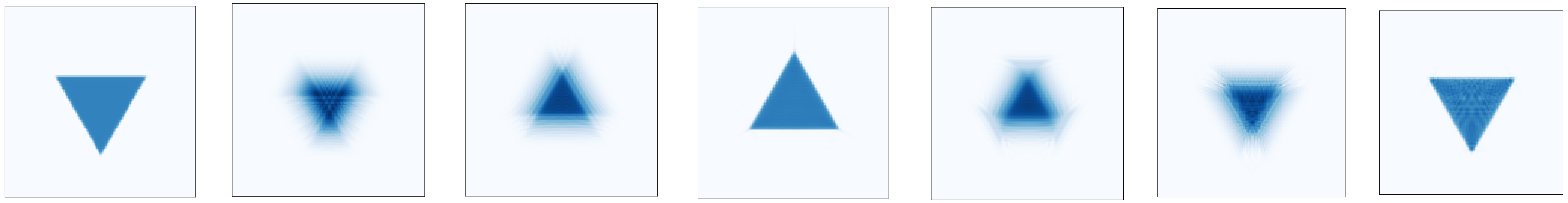
Saint-Jalm et al,
Phys. Rev. X 9, 021035 (2019)

Experimentally, in a harmonic trap of frequency ω:



period $T/2$ with $T = 2\pi/\omega$

Numerically, solution of the Gross-Pitaevskii equation on a 1024x1024 grid:



Initial state $|\psi_i\rangle$: uniform filling of the triangle

Overlap with wave function at $T/2$: $|\langle\psi_i|\psi_f\rangle| > 0.995$

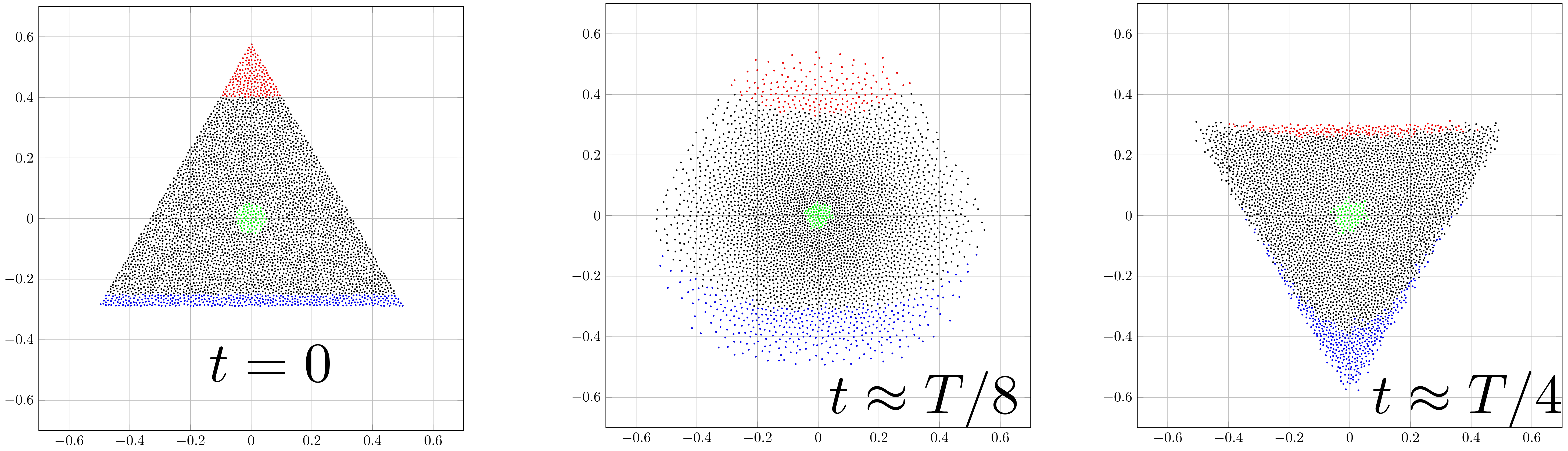
Does not seem to occur for any other polygonal shape!

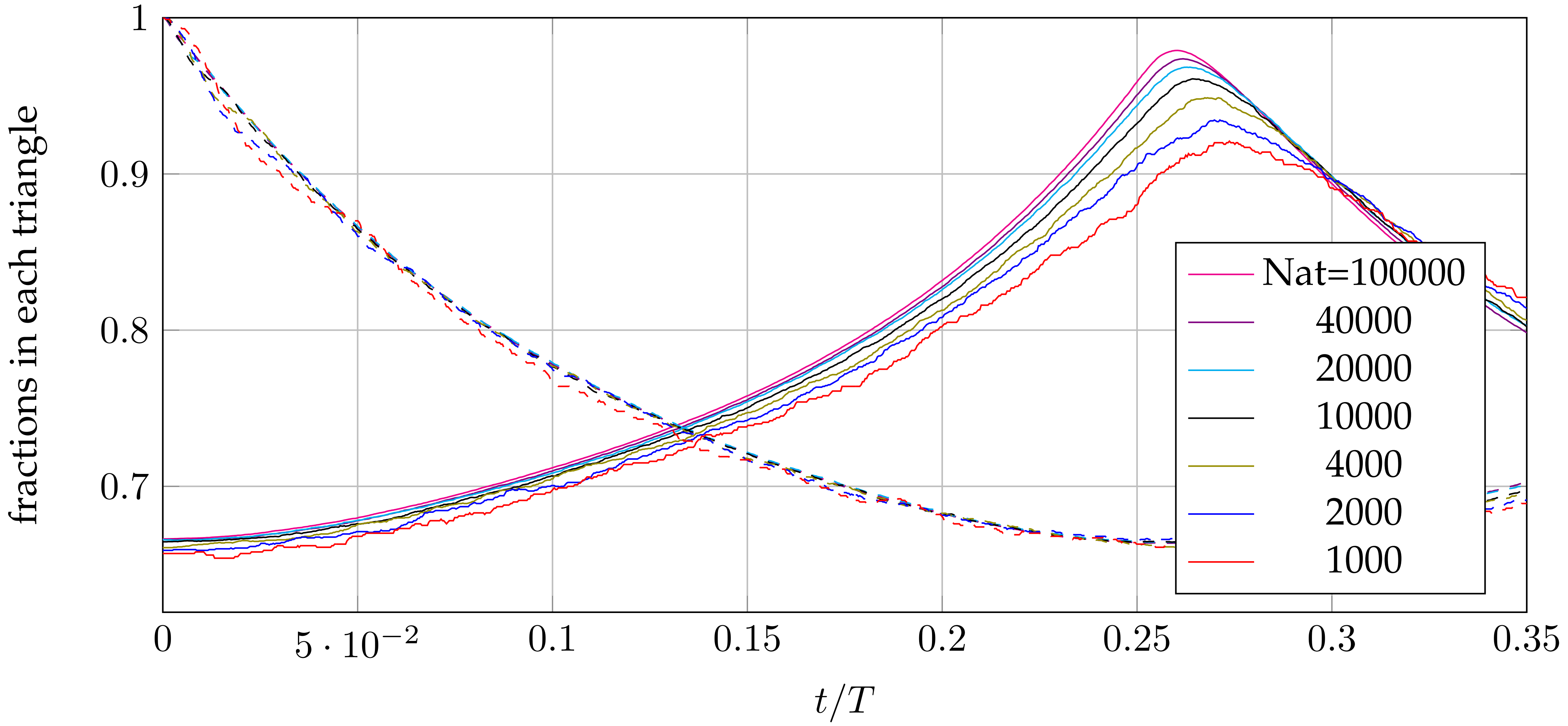
Do such breathers also show up for other 2D systems with SO(2,1) symmetry?

A simple test: Classical particles interacting with $V(r) \propto \dfrac{1}{r^2}$ potential $\qquad V(r/\lambda) = \lambda^2 V(r)$

Simulation with 4000 particles

$$\boldsymbol{v}_j(0) = 0$$





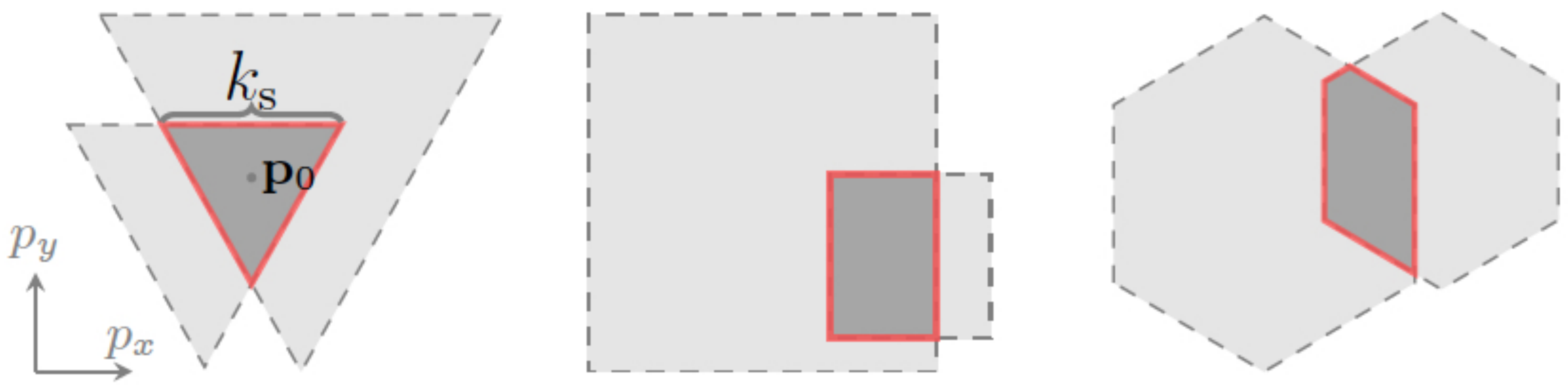
Two recent theoretical insights

Shi, Gao & Zhai, Phys. Rev. X 11 041031 (2021): "Ideal-Gas Approach to Hydrodynamics"

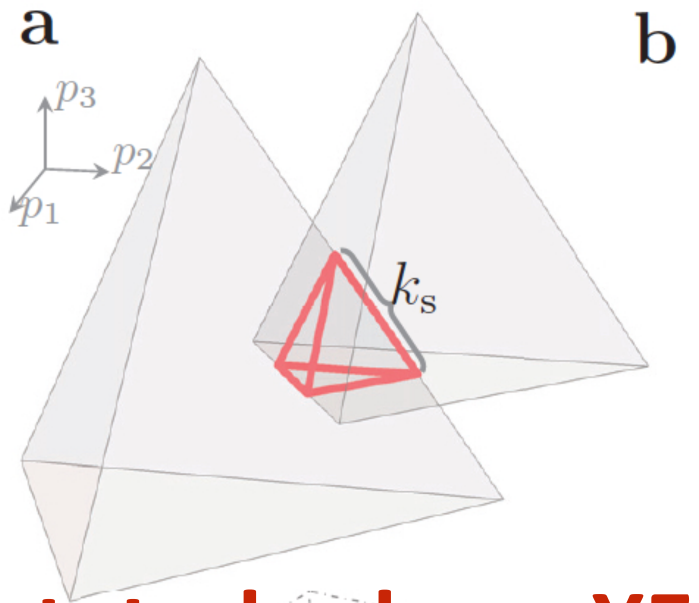
"There exist situations that the solution to a class of interacting hydrodynamic equations with certain initial conditions can be exactly constructed from the dynamics of noninteracting ideal gases"

In the proof, scale invariance appears as a necessary, but not sufficient, condition

Specific shapes : the overlap area of two homothetic equilateral triangles is always of the same shape



YES NO NO



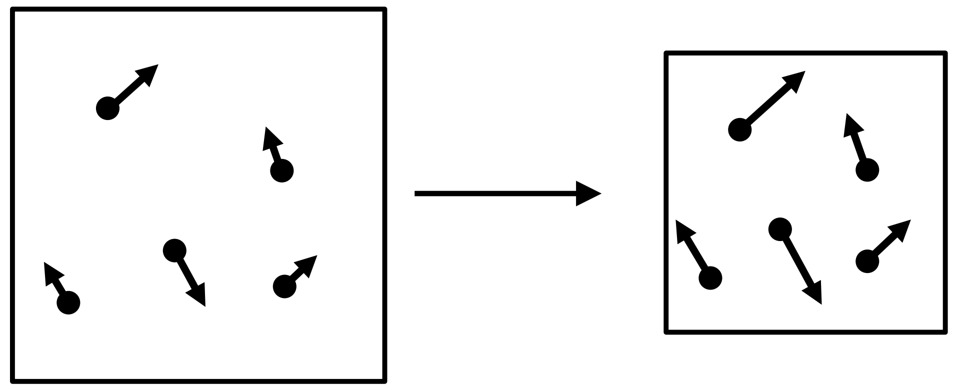
3D tetrahedron: YES

Olshanii et al, SciPost Phys. 10, 114 (2021): "Triangular Gross-Pitaevskii breathers and Damski-Chandrasekhar shock waves"

The shock wave created by the initial density jump does not induce further catastrophes in the hydrodynamic equations

Summary

Conformal invariance: example of a dynamical (or hidden) symmetry

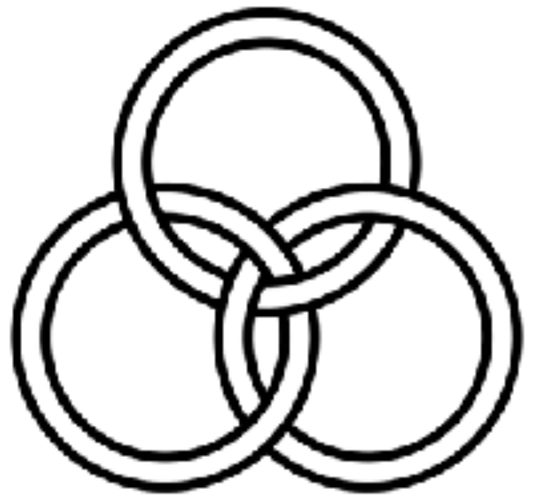


Transformations that leaves the equations of motion invariant

Valid either at the quantum-field level (3D unitary gas) or at the classical-field level (2D Bose gas)

In the latter case, provides an example of a quantum anomaly in low-energy physics

A situation valid in any dimension: the $1/r^2$ interaction potential



Can this potential be simulated for a many-body quantum gas, besides the now well-understood Efimov effect?

Thank you!

Breaking the SO(2,1) symmetry with a quantum anomaly

Hammer & Son (2004): Going beyond the classical field analysis based on the Gross-Pitaevskii energy functional

Introduction of a short-distance (i.e. UV) cutoff at $r \sim R_{\rm vdW}$ (van der Waals length : nanometer size)

→ There exists a stable solution of size σ_N for any value of the atom number N

Geometric scaling: $\dfrac{\sigma_N}{\sigma_{N+1}} \sim 3$

$$\sigma_{1020} \sim 10^{-9}\ \sigma_{1000}$$
$$\sigma_{980} \sim 10^{+9}\ \sigma_{1000}$$

In practice, for an interaction strength $|g| \ll 1$, the predicted value for σ_N is physically reasonable only for $|N - N_{\rm Townes}| \sim$ a few units

→ In the strongly interacting case $|g| \sim 1$, a realistic droplet size would be achieved with only a few atoms and one could observe the predicted scaling of σ_N with N